The Idea of a Grammatical Calculus

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Hello, Everyone! Good morning, good afternoon and good evening!

Thanks for coming! To the organizers of the conference, especially to the tireless Professor Toufik Mansour, I would like to say

I am honored and humbled to happen to be on the list of speakers and to be monitored by such a distinguished audience.

Where should I begin? I asked myself. The King in the Wonderland said, begin at the beginning. But I am afraid that I might end up with the beginning as well. Anyway, let us get started with the idea of a programming language or a formal language, which might sound irrelevant and even bonkers. But please bear with me for a second.

As one can imagine, the fundamental issue of a programming language is to regulate every sentence so that there is no ambiguity whatsoever. You simply cannot say one thing and mean another, and the machine does not have to read the tone between the lines, like in a diplomatic statement. Then the salvation came from the revolutionary concept, namely, the idea of a context-free grammar.

To make a long story short, a context-free grammar consists of a set of production rules. Each rule plays his own part regardless of the context, just like you do your duty and you do not really care what other people think about you, contrary to the real life scenarios.

While the grammars we are concerned with are not exactly the same as in practical use, due to completely different purposes, their constructions fall into the same framework, at least in theory.

Connection to the Umbral Calculus

Is it far fetched to relate to context-free grammars? You may ask. I can't answer in one word. Let us consider the derivative property:

$$D(fg) = D(f)g + fD(g).$$

Does this ring a bell to you?

Observation: If you stare at it for some time, you may wonder if a derivative operator D is context-free in the sense that when D is taking an action on f, it is none of the business of g, and vice versa.

As an after event wisdom, it is not insane to claim that the ideology of context-free grammars can be traced back to the concept of differential operators. Even if it were true, it would be a pointless and worthless truth. But there is something to it, when it comes to certain situations regarding combinatorial enumeration, for some combinatorial structures are intrinsically context-free.

Such context-free properties may look inconspicuous and some may have been taken for granted, but they may have the potential to accomplish formidable missions like secret agents. This is truly the embarkation point of our journey.

Instead of projecting decorated theorems and dwelling on the rigorous derivations, I have a feeling that it's probably more authentic, rather, to be less formal, and just to have a chat about some basic ideas and intuitions that are not touched upon in the technical publications.

Let us be a little more serious. Consider the grammar $f \to f$. What is $D(f^{-1}) =$? Answer:

$$\boxed{D(f^{-1}) = -f^{-1}.}$$

Since $ff^{-1} = 1$, The derivative property demands that

$$D(f)f^{-1} + fD(f^{-1}) = 0,$$

which yields $D(f^{-1}) = -f^{-1}$. On the other hand, in view of the chain rule, we get the same result.

Look at this lovely minus sign here! It tells that the formal derivative D behaves like a sign-reversing operator. And then?

This property reminds us of inverse relations. Let's turn to the classical example of the umbral calculus. Rota's magical interpretation of the old-fashioned symbolic method by using linear functionals demystifies the illusion of lifting a subscript of a_n to a superscript of a^n, that is,

$$a_n = L(x^n).$$

A benchmarking example is the following inverse pair:

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k \quad \Leftrightarrow \quad b_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k.$$

Does the above grammar have anything to do with the umbral calculus?

– Yes, it does! Let's see how it works.

Treat $b_0, b_1, b_2, \ldots$ as a sequence of symbols, or variables, or indeterminates bearing no meaning. Define the grammar as

$$G = \{f \to f,\ b_i \to b_{i+1} \mid i = 0, 1, 2, \ldots\}.$$

Let D denote the formal derivative with respect to G. Suppose that the relation on the left holds. By the Leibniz formula, we get

$$a_n = \sum_{k=0}^{n} \binom{n}{k} b_k = f^{-1} D^n (f b_0).$$

Thus,

$$b_n = D^n(b_0) = D^n(f^{-1} f b_0),$$

which yields

$$\sum_{k=0}^{n} \binom{n}{k} D^{n-k}(f^{-1}) D^k(f b_0) = \sum_{k=0} (-1)^{n-k} \binom{n}{k} a_k,$$

as required.

The above reasoning may be related to something elusively remarked by Professor Doron Zeilberger in his review article on the umbral calculus. I must confess that I am ignorant and likely going to be even more so of the early history on the subject. Nevertheless, we gotta move on.

The Eulerian Polynomials

As the next example, let's consider the grammar

$$G = \{x \to xy, \quad y \to xy\}. \tag{1}$$

This is a grammar related to the Eulerian polynomials involving the number of descents of a permutation. It was formally presented by Dumont. In fact, this grammar led us to the notion of a grammatical labeling [C.-Fu, 2017].

Remark. The two variables x and y are needed in order to perform the grammatical calculus.

First, we encounter the two choices of notation. On one hand, a grammar is represented by a set of substitution rules. On the other hand, it can be encoded by a differential operator:

$$D = xy\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right), \tag{2}$$

which is a *creation operator*, rather than an *annihilation operator*.

Dumont argued that grammar notation has certain advantages. On the other hand, he did not elaborate on the advantage of the operator notation in doing experiments with computer algebra.

The above operator D can be perceived as a derivative defined by

$$D(x) = xy, \quad D(y) = xy.$$

Thus we have

$$D^2(x) = D(xy) = D(x)y + xD(y) = xy^2 + x^2y,$$

$$D(xy^2) = D(x)y^2 + xD(y^2) = xy^3 + 2xyD(y) = xy^3 + 2x^2y^2,$$

$$D(x^2y) = D(x^2)y + x^2D(y) = 2xyD(x) + x^3y = 2x^2y^2 + x^3y.$$

Hence we get

$$D^3(x) = xy^3 + 4x^2y^2 + x^3y.$$

For $n \geq 1$, the bivariate Eulerian polynomials are defined by

$$A_n(x, y) = \sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} y^{\mathrm{asc}(\sigma)},$$

where σ ranges over all permutations of $[n] = \{1, 2, \ldots, n\}$, and $\mathrm{des}(\sigma)$ and $\mathrm{asc}(\sigma)$ are defined with the understanding that σ is patched a zero both at the beginning and at the end.

Theorem (Dumont)

For $n \geq 1$, we have

$$A_n(x, y) = D^n(x).$$

As will be seen, the above relation has a proof without words.

A grammatical calculus for the Eulerian polynomials

Let w be a formal power series, or a Laurent series on the variables of a grammar G, and let D be the formal derivative relative to G. Define the generating function

$$\mathrm{Gen}(w,t) = \sum_{n=0}^{\infty} D^n(w)\frac{t^n}{n!}.$$

Generating function of $A_n(x,y)$

Set $A_0(x,y) = x$. Then we have

$$\mathrm{Gen}(x,t) = \sum_{n=0}^{\infty} A_n(x,y)\frac{t^n}{n!} = \frac{x-y}{1-x^{-1}y\,e^{(x-y)t}}.$$

Since $D(x) = xy$ and $D(y) = xy$, we have

$$D(x^{-1}) = -x^{-2}D(x) = -x^{-1}y,$$

$$D(\underline{x^{-1}y}) = -x^{-1}y^2 + y = (x - y)\,\underline{x^{-1}y}.$$

The factor $x - y$ is something special to D. According to the grammar G, both x and y are substituted by xy. But x and y are supposed to be different variables. What is really going on here?

Answer: $\underline{x - y}$ is a constant with respect to D.

At first glance, one would not look upon $x - y$ as a constant as one is used to what is meant to be a constant. However, this individual D has his own judgement, because what matters for D is merely whether

$$D(x - y) = 0.$$

This constant property makes it possible to deduce a simple expression of $D^n(x^{-1})$ for all $n \geq 1$,

$$\boxed{D^n(\underline{x^{-1}y}) = (x - y)^n \, \underline{x^{-1}y}.}$$

Consequently, we are led to the generating function of x^{-1}.

Since

$$\mathrm{Gen}(x,t)\,\mathrm{Gen}(x^{-1},t) = 1,$$

we instantly arrive at the generating function of the bivariate Eulerian polynomials.

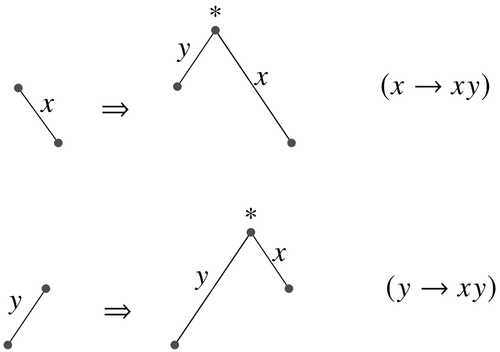
Are you convinced that this is all the computation we need? Could it have been simpler? It might seem a little odd that this innocent looking constant $x - y$ does play a crucial part beyond expectation!

A grammatical labeling for the Eulerian polynomials

As promised, we now give a grammatical labeling of permutations reflecting the number of descents. Given $\sigma = \sigma_1\sigma_2\cdots\sigma_n \in S_n$, assume that $\sigma_0 = \sigma_{n+1} = 0$. For $0 \leq i \leq n$, we label the position between σ_i and σ_{i+1} by x if i is a descent, and by y if i is an ascent, for example,

$$0\,y\,3\,y\,5\,x\,2\,y\,4\,x\,1\,y\,6\,x\,0.$$

This labeling scheme is essentially the same the descent word on $\{U, D\}$ or the ab-index of the symmetric group in noncommutative variables a and b, see [R. Stanley, Longest alternating subsequences of permutations] and [R. Stanley, A survey of alternating permutations].



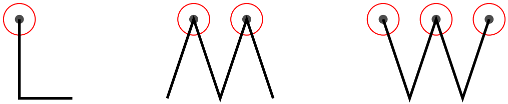
This a proof without words! The picture exhibits what happens when $n + 1$, signified by $*$, is inserted to a permutation of $[n]$. The update is apparently a local or a context-free operation.

The left peak polynomials

What is a peak? Here is a line that is perhaps not entirely unfit:

But what for permutations? Well, first of all, special attention has to be paid to the first position and the last position. Thereby we are faced with three types of peaks: left peaks, interior peaks and exterior peaks. By symmetry, right peaks can be treated equally as left peaks.

There have been various and sporadic notations for the three or four kinds. We found it hard to set our mind to which to follow. We felt relieved, at least to ourselves, that we came up with a possible solution, and we hope that they are acceptable to you as well unless there are preferred alternatives. Here is our proposal: L for left peaks, M for interior peaks, and W for exterior peaks. Recall that as far as a left peak is concerned, a zero is patched to the beginning of a permutation, and other types of peaks are defined in the same fashion. The following illustration reveals why:



Incidentally, M is reminiscent of "middle", and W of "wide", in contrast with "interior" and "exterior".

The bivariate left peak polynomials are defined by

$$L_n(x, y) \quad = \quad \sum_{k=0}^{\lfloor n/2 \rfloor} L(n, k) x^{2k+1} y^{n-2k}, \tag{3}$$

where $L(n, k)$ denotes the number of permutations of $[n]$ with k left peaks. For $n = 0$, we define $L_0(x, y) = x$.

The interior peak polynomials $M_n(x, y)$ are defined in the same manner.

The grammar for the left peak polynomials $L_n(x)$ was discovered by Ma via a recurrence relation and independently by C.-Fu in terms of a grammatical labeling. Ma further noticed that this grammar can also be employed to generate the interior peak polynomials. Here is the grammar

$$G = \{x \to xy, \;\; y \to x^2\}. \tag{4}$$

Let D be the formal derivative with respect to the grammar G.

The grammatical interpretation

For $n \geq 0$, we have

$$D^n(x) = L_n(x, y).$$

A grammatical calculus for the Gessel formula

In the single variable version, David-Barton established a system of partial differential equations involving $L(x,t)$ and $M(x,t)$, and found a solution requiring one more step of integration. An explicit expression of $L(x,t)$ was given by Gessel. Alternative proofs of the formulas for $L(x,t)$ and $M(x,t)$ have been given by Zhuang.

The bivariate form of the Gessel formula

$$\sum_{n=0}^{\infty} L_n(x,y)\frac{t^n}{n!} = \frac{x\sqrt{y^2-x^2}}{\sqrt{y^2-x^2}\cosh(t\sqrt{y^2-x^2}) - y\sinh(t\sqrt{y^2-x^2})}.$$

Appealing to the grammar, we find that $y^2 - x^2$ is a constant, since

$$D(x^2) = D(y^2) = 2x^2 y.$$

Now we aim to compute $D^n(x^{-1})$ for $n \geq 1$. Starting with

$$D(x^{-1}) = -x^{-1} y, \quad D^2(x^{-1}) = x^{-1}(y^2 - x^2),$$

the following rhythm emerges. For $n \geq 0$,

$$\begin{aligned} D^{2n}(x^{-1}) &= x^{-1}(y^2 - x^2)^n, \\ D^{2n+1}(x^{-1}) &= -x^{-1} y (y^2 - x^2)^n. \end{aligned}$$

Taking the parity into account, we deduce that

$$\sum_{n=0}^{\infty} D^{2n}(x^{-1})\frac{t^{2n}}{(2n)!} = x^{-1}\cosh\left(t\sqrt{y^2-x^2}\right),$$

$$\sum_{n=0}^{\infty} D^{2n+1}(x^{-1})\frac{t^{2n+1}}{(2n+1)!} = -\frac{x^{-1}y}{\sqrt{y^2-x^2}}\sinh\left(t\sqrt{y^2-x^2}\right).$$

Now, we see how the hyperbolic functions come on the scene.

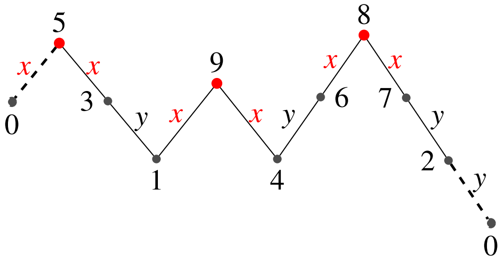
For the rest, Bob is your uncle, as we say in UK. Voila!

I hope that I didn't miss anything. Actually, it took us quite a while to reach this point largely because we did not anticipate such a short cut. We were nearly there, but we missed the target in the previous pathetic attempts [C.-Fu-2017].

A grammatical labeling for left peaks

To leave no doubt about the rigor of the grammatical apparatus, we are obliged to provide a grammatical labeling. Once this step is finished, before you know it, you may have a proof without words.

The procedure is straightforward. Given a permutation σ of $[n]$, display it in the style of a path, more or less the same as the mountain range view [T.K. Petersen, Eulerian Numbers]. Then label the steps of the path. Wherever there is a peak, there are two x's. The remaining steps are labeled with y.



The substitution rules are readily seen to be context-free, because the insertion of $n+1$ into a permutation of $[n]$ only makes a local impact on change of the descents. This property is precisely what is needed to put the grammatical calculus on a firm footing, without thinking of the notation $A(n, k)$ for the Eulerian numbers.

The idea of transformations of grammars can be mobilized to prove combinatorial identities. Here is one stone for two birds.

Theorem (C.-Fu)

Setting

$$x = \sqrt{\bar{x}\bar{y}}, \quad y = (\bar{x} + \bar{y})/2,$$

then the grammar

$$G = \{\bar{x} \rightarrow \bar{x}\bar{y}, \ \ \bar{y} \rightarrow \bar{x}\bar{y}\}$$

is transformed into the grammar

$$G = \{x \rightarrow xy, \quad y \rightarrow x^2\}.$$

Let D_1 be the formal derivative with respect to the first grammar and let D_2 be the formal derivative with respect to the second grammar.

Resorting to the transformation, we have

$$D_1(x) = xy, \quad D_1(y) = x^2.$$

Since $D_2(\bar{x}\bar{y}) = (\bar{x} + \bar{y})\bar{x}\bar{y}$, we infer that

$$D_1(x) = D_2(\sqrt{\bar{x}\bar{y}}) = \frac{D_2(\bar{x}\bar{y})}{2\sqrt{\bar{x}\bar{y}}} = \sqrt{\bar{x}\bar{y}}\,\frac{\bar{x} + \bar{y}}{2} = xy$$

and

$$D_1(y) = D_2\left(\frac{\bar{x} + \bar{y}}{2}\right) = \bar{x}\bar{y} = y^2,$$

as required.

The Petersen identity

Theorem (The bivariate form of the Petersen identity)

For $n \geq 1$, we have

$$L_n\left(\sqrt{xy}, \frac{x+y}{2}\right) = \sqrt{xy^{-1}} \sum_{k=0}^{n} \binom{n}{k} A_k(x,y) \frac{(y-x)^{n-k}}{2^{n-k}}.$$

As long as we are furnished with the transformation on the two grammars, it is simply a matter of formality to bring out the connection between the Eulerian polynomials and the left peak polynomials, due to Petersen.

By the transformation of grammars, we obtain

$$D^n(x) = D^n(\sqrt{\bar{x}\bar{y}}) = D^n\left(\bar{x}\sqrt{\bar{x}^{-1}\bar{y}}\right).$$

It follows that

$$L_n(x, y) = \sum_{k=0}^{n} \binom{n}{k} D^k(\bar{x}) D^{n-k}\left(\sqrt{\bar{x}^{-1}\bar{y}}\right).$$

But $D^k(\bar{x}) = A_k(\bar{x}, \bar{y})$ and the constant property of $\bar{x} - \bar{y}$ ensures that

$$D^{n-k}\left(\sqrt{\bar{x}^{-1}\bar{y}}\right) = \frac{1}{2^{n-k}}(\bar{x}\bar{y}^{-1})^{\frac{1}{2}}(\bar{y} - \bar{x})^{n-k},$$

we obtain the Petersen identity by switching $\bar{x}$ and $\bar{y}$ back to x and y.

The interior peak polynomials

Adopting the notation $M(n,k)$ for the number of permutations of $[n]$ with k interior peaks. The bivariate interior peak polynomials are defined by $M_0(x,y)=y$ and for $n\geq 1$,

$$M_n(x,y) = \sum_{k=0}^{\lfloor n/2 \rfloor} M(n,k)x^{2k+1}y^{n-2k}. \tag{5}$$

Let D be the formal derivative with respect to the grammar G.

The grammatical interpretation of Ma

For $n\geq 0$, we have

$$D^n(y)=M_n(x,y).$$

A grammatical labeling for interior peaks is given by [C.-Fu, 2023].

The Stembridge identity

The above transformation of grammar also applies to the Stembridge identity on the Eulerian polynomials and the interior peak polynomials, which was derived in his theory of enriched P-partitions. Brändén rediscovered this identity utilizing the “modified Foata-Strehl action”.

Theorem (The bivariate form of the Stembridge identity)

For $n \geq 1$, we have

$$A_n(x, y) = M_n\left(\sqrt{xy}, \frac{x+y}{2}\right).$$

Let D_1 and D_2 be given as before. We have

$$A_n(\bar{x}, \bar{y}) = D_2^n(\bar{x}) = D_2^{n-1}(\bar{x}\bar{y}) = D_1^{n-1}(x^2) = D^n(y) = M_n(x, y).$$

On the other hand, by the change of variables, we have

$$M_n(x, y) = M_n\left(\sqrt{\bar{x}\bar{y}}, \frac{\bar{x} + \bar{y}}{2}\right).$$

Thus we are led to the Stembridge identity by renaming $\bar{x}$ and $\bar{y}$.

Setting $y = 1$, we recover the Stembridge identity in the original form.

$$\sum_{\sigma \in S_n} x^{\mathrm{des}(\sigma)} = \left(\frac{1 + x}{2}\right)^{n-1} \sum_{k \geq 0} M(n, k) \left(\frac{4x}{(1 + x)^2}\right)^k. \tag{6}$$

The Elizalde-Noy formula

For a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$, an index i $(3 \leq n)$ is called a proper double descent if $\sigma_{i-2} > \sigma_{i-1} > \sigma_i$. Let $U_n(y)$ be the generating function for the number of permutations of $[n]$ with k proper double descent, and let $U(y,t)$ be the exponential generating function of $U_n(y)$.

The Elizalde-Noy formula

The generating function $U(y,t)$ equals

$$\frac{2\sqrt{(y-1)(y+3)}\, e^{t/2\cdot(1-y+\sqrt{(y-1)(y+3)})}}{1+y+\sqrt{(y-1)(y+3)}-(1+y-\sqrt{(y-1)(y+3)})\, e^{t\sqrt{(y-1)(y+3)}}}.$$

In a more general setting, a grammatical calculus has been carried out by Fu (2018). As a refinement of the grammar for left peaks, Fu came up with the grammar

$$G = \{x \to xy,\ y \to xz,\ z \to zw,\ w \to xz\}.$$

It is not clear whether this grammar measures up to the computation of the joint distribution of the number of left peaks and the number of proper double descents. The formula of Fu turns out to be a unification of these of Gessel and Elizalde-Noy, for $y = 1$ and $x = 1$, respectively:

$$\frac{2\sqrt{(1+y)^2-4x}\, e^{t/2\cdot(1-y+\sqrt{(1+y)^2-4x})}}{1+y+\sqrt{(1+y)^2-4x}-(1+y-\sqrt{(1+y)^2-4x})\, e^{t\sqrt{(1+y)^2-4x}}}.$$

That's quite a remarkable feature.

The γ-positivity

With the aid of the grammar, the γ-positivity of the Eulerian polynomials becomes transparent. Ma-Ma-Yeh made a change of variables:

$$u = xy, \quad v = x + y.$$

Since

$$D(u) = D(xy) = xy^2 + x^2y = uv, \quad D(v) = D(x + y) = 2xy = 2u,$$

the grammar G is transformed into a new grammar

$$H = \{u \to uv,\ v \to 2u\}.$$

This new grammar is obviously nonnegative, as we see nothing negative. Thus there is no question about the γ-positivity (Foata-Schüzenberger). Moreover, the transformed grammar lends a new perspective on the combinatorial interpretation of the γ-coefficients.

Based on the grammar, we find that the increasing binary tree and the increasing plane tree settings seem to be more convenient than the permutation setting to deal with the γ-positivity.

To be more precise, the Foata-Strehl group action and its modified versions can be better visualized on trees, see [C.-Fu-Yan, 2023].

This argument can be pushed forward to deduce the e-positivity of the trivariate generating function $C_n(x, y, z)$ associated with three statistics of Stirling permutations introduced by Gessel and Stanley [C.-Fu-2022]. These polynomials were introduced by Dumont, and are called the trivariate second-order Eulerian polynomials. They are generated by the grammar

$$\{x \to xyz, \quad y \to xyz, \quad z \to xyz\},$$

as presented by Dumont (1980). A refined recurrence was given by Haglund and Visontai (2012).

In addition to the numbers of ascents and descents, Bóna independently introduced the notion of a plateau in a Stirling permutation and showed that all the three statistics obey the same distribution.

Here is an interesting story. Ma stumbled on a paper of Dumont in French, and thanks to the machine translation, he realized that the notion of a plateau in a Stirling permutation defined by Bóna coincides with the statistic defined by Dumont under the name of a repetition (in French, of course).

It might be possible that grammars can be helpful in the further studies of positivity questions associated with combinatorial structures admitting context-free constructions.

Up-down runs of permutations

It seems to be an ever-lasting topic in enumerative combinatorics to inspect the number of up-down runs of a permutation. Roughly speaking, the number of up-down runs equals the number of turning points. In particular, alternating permutations are a special class of permutations with a fixed number of up-down runs. Given a permutation $\sigma = \sigma_1\sigma_2\cdots\sigma_n$ of $[n]$, assume $\sigma_0 = 0$, that is, a zero is patched at the beginning. An up-down run of σ is a maximal segment that is either increasing or decreasing. For example, the permutation 3 7 5 8 6 1 4 9 2 has six up-down runs:

$$0\,3\,7\,,\ 7\,5,\ 5\,8,\ 8\,6\,1,\ 1\,4\,9,\ 9\,2.$$

Here is the grammar for up-down runs found by Ma:

$$G = \{a \to ax, \;\; x \to xy, \;\; y \to x^2\},$$

As expected, one can carry out the grammatical calculus to derive the generating function, denoted by $\Lambda(x,t)$, in our proposed notation. This has been done by [C.-Fu, 2023]. In the traditional way, as the first step, one tries to establish a recurrence relation by a “clever counting”, as Professor Stanley would have put it. Even if this is done, we are left with the tasks of setting up and solving an equation.

Here we don't intend to compare the grammatical fantasy with the classical strategies, and I am inclined to stick to the belief that

— Sometimes the old ways are the best. — Skyfall

Grammar assisted bijections

Apart from the grammatical calculus, we find that a grammar can be helpful in constructing bijections and it can serve as a guide to discover something new.

It is well-known that the Euler number E_{2n} can be interpreted in terms of alternating permutations and even increasing trees. A.G. Kuznetsov, I.M. Pak and A.E. Postnikov have given a bijection. We [C.-Fu, 2017] extended this bijection to incorporate the number of left peaks. Our construction is somehow rather technical and we find it more and more difficult to remember.

With the grammar in hand, we were much better off and were able to build a natural correspondence which maps several permutation statistics involving up-down runs and peaks to tree statistics involving the parity of the number of children of each vertex. The theorems are eliminated here.

The trick can be described as a reflection principle. Hopefully, we have hunted a most wanted correspondence between permutations and increasing trees that can be restricted to alternating permutations and even increasing trees.

In summary, if a suitable grammar is at disposal and if we were lucky, it may happen that this very grammar can solve everything all at once.

Dominique Dumont

Professor Dominique Dumont was a great advocate of grammars and made a number of remarkable discoveries. What I found the most striking is the grammar for the Ramanujan polynomials, in connection with Shor's refinement of the classical formula of Cayley.

I never had the pleasure of meeting Professor Dominique Dumont. To show my appreciation, I expressed my wish to invite him for a visit. He kindly accepted the invitation and even made a plan. But, sadly, it was a profound regret that he didn't make it. Now, all there is to say is that his mathematics remains to bring me joy and happiness.

The Ramanujan polynomials

When speaking of Ramanujan, it is hard to connect him to the enumeration of trees. He may not have been aware of what he had done in combinatorial terms. In the monumental edition of Ramanujan's Second Notebooks, compiled by B. C. Berndt, R. J. Evans and B. M. Wilson, the Ramanujan polynomials have ultimately found a permanent residence accessible to combinatorialists.

J. Zeng should be attributed for observing the connection between the Ramanujan polynomials and Shor's interpretation.

This relationship brings the grammatical calculus to the territory of rooted trees and Abel-type identities. We shall demonstrate how a grammar can be employed to deal with an Abel-type identity arising from the theory of machine learning.

Once we have the grammar in place, the identity turns out to be a disguise of the Leibniz formula. This conjectural identity has attracted quite a few combinatorialists to try their hands from different angles.

To my surprise, this is not the end of the story. One day, I was contacted by my friend Professor J. Zhou at Tsinghua University, a geometer, about the Ramanujan polynomials and something strange to me, such as the orbifold Euler characteristics of the moduli spaces of stable curves. While being astonished, I responded with the question of how he could have the faintest idea of something in our trade. His answer was comforting: He tried his luck with the Online Encyclopedia of Integer Sequences (OEIS), and the search result put us on a common ground. Later on, it came to my mind that this was probably a hint that I should make a donation to the Foundation.

A question

As to the Ramanujan polynomials, Shor asked a question of giving a combinatorial interpretation of a recurrence relation. Victor Guo and I worked out a rather tedious bijection. We knew that it was not the one we were looking for. Then Guo went on to find a simpler construction. Yet, I still prefer to see something simple, not just simpler.

The Ramanujan polynomials $R_n(q)$

Definition

Set $R_1(q) = 1$. For $n \geq 1$, define

$$R_{n+1}(q) = n(1+q)R_n(q) + q^2 R_n'(q), \tag{7}$$

Of course, they are not just motivated by a recurrence relation. For example,

$$R_1(q) = 1, \quad R_2(q) = 1 + q, \quad R_3(q) = 3q^2 + 4q + 2,$$

$$R_4(q) = 15q^3 + 25q^2 + 18q + 6.$$

Positivity properties

- [C.-Wang-Yang, 2011] The Ramanujan polynomials $R_n(q)$ are strongly q-log-convex, that is,

$$R_{m-1}(q)R_{n+1}(q) \geq_q R_m(q)R_n(q)$$

for all $n \geq m \geq 1$, where $f(q) \geq_q g(q)$ means that the difference $f(q) - g(q)$ has nonnegative coefficients as a polynomial of q.

- [A.D. Sokal, 2021] The Hankel matrix $(R_{i+j}(q))_{i,j\geq 0}$ associated with the Ramanujan polynomials $R_n(q)$ is coefficientwise totally positive (that is, all its minors of this Hankel matrix are polynomials with nonnegative coefficients.)

Shor's refinement of Cayley's formula

Theorem (Shor)

Let T_n denote the set of rooted trees on $[n]$, and let $\mathrm{imp}(T)$ denote the number of improper edges of a rooted tree T. Then

$$R_n(q) = \sum_{T \in T_n} q^{\mathrm{imp}(T)}$$

An edge (u, v) with u being the parent and v being a child is said to be improper if there exists a descendant of v that is smaller than u, under the assumption that any vertex of T is considered as a descendant of itself, or equivalently, (u, v) is said to be proper if all the vertices in the subtree rooted at v are greater than u.

The grammar of Dumont and Ramanonjisoa

Dumont and Ramamonjisoa (1996) introduced the following grammar for the enumeration of rooted trees with a given number of improper edges:

$$G = \{A \to A^3 S, \quad S \to AS^2\}.$$

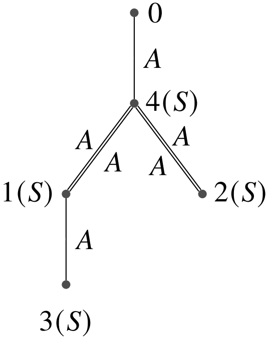
Let D denote the formal derivative with respect to the grammar G.

Theorem (Dumont and Ramanonjisoa)

For $n \geq 1$,

$$D^{n-1}(AS) = A^n S^n \sum_{T \in T_n} A^{\mathrm{imp}(T)} = A^n S^n R_n(q).$$

Dumont and Ramamonjisoa gave a proof via the recurrence relation of Shor. We found a combinatorial interpretation in terms of a grammatical labeling [C.-Yang, 2021]. Instead of elaborating on the definition of the grammatical labeling, we only offer a snapshot to show how it looks, where an improper edge is represented by a double edge:



The Lacasee identity

The Dumont-Ramamonjisoa grammar reveals that rooted trees can be recursively constructed via local (or context-free) operations. The grammatical labeling can be thought as a description of the concrete procedure.

Next, we present an application of the Dumont-Ramamonjisoa grammar (C.-Yang, 2021).

Theorem (Lacasse)

$$n^{n+1} = \sum_{k=1}^{n} \sum_{k=0}^{n-j} \binom{n}{j} \binom{n-j}{k} j^j k^k (n-j-k)^{n-j-k}.$$

This was conjectured by Lacasse in the study of the theory of machine learning. Since then, several proofs have been found.

- Y. Sun (2013), using the umbral calculus;
- Prodinger (2013), using Cauchy's integral formula;
- Chen, Peng and Yang (2013), a combinatorial interpretation;
- Gessel (2016), using the Lagrange inversion formula.

Rewrite it as

$$n^n = \sum_{j=1}^{n} \sum_{k=0}^{n-j} \binom{n-1}{j-1} \binom{n-j}{k} j^{j-1} k^k (n-k-j)^{n-k-j}.$$

Since $D(A) = A^3 S$, we have the convolution formula

$$D^n(A) = D^{n-1}(A^3 S) = \sum_{i+j+k=n} \binom{n-1}{i, j-1, k} D^i(A) D^{j-1}(AS) D^k(A),$$

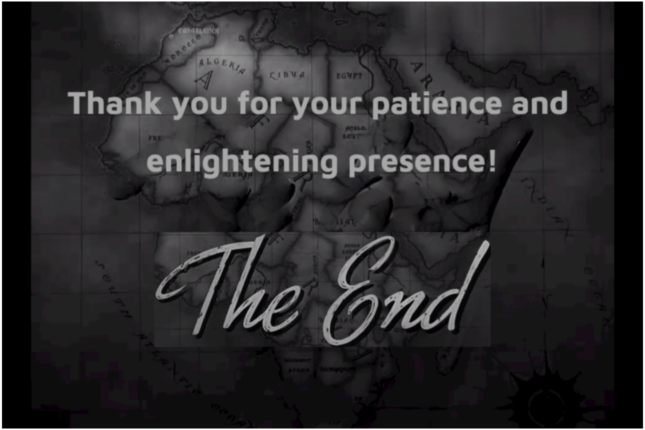
in accordance with what we are seeking, owing to the following facts

$$\begin{aligned} D^{n-1}(AS)|_{A=S=1} &= n^{n-1}, \\ D^{n-1}(A)|_{A=S=1} &= n^n. \end{aligned}$$

It is probably the time for myself to ask a question. Usually, at certain point during my lecture I would ask my students whether they wanted to hear more. I always got a “No” without “Thank you”.

The late Professor S. S. Chern once said something like the highlight of a meeting comes at the moment of making an announcement that everyone is looking forward to.

Now, here it is.


Thank you for your patience and
enlightening presence!
The End